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SCHEDULING A THREE SUPPLY SHIP REPLENISHMENT OPERATION

by

John McKee Huling, Jr.

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THESIS

SCHEDULING A THREE SUPPLY SHIP

REPLENISHMENT OPERATION

by

John McKee Huling, Jr.

October 1969

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Scheduling a Three Supply Ship Replenishment Operation

by

John McKee Huling, Jr.
Lieutenant, United States Navy
B.S., United States Naval Academy, 1962

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

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ABSTRACT

The operation of replenishment at sea is investigated for three supply ships and L combatants using queueing theory concepts and a random walk model in three dimensions. The distribution for total replenishment time given the initial number of combatants to be replenished by each supply ship and a specified cyclic order of replenishment is expressed in terms of its Laplace transform under the assumption of independent, exponential service times for each supply vessel. A method for counting all possible sequences of replenishment is not found, but some preliminary counting techniques are developed which may be useful in its eventual determination.

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I. INTRODUCTION

A major aspect of modern naval policy is the stationing of task forces throughout the world, and the supply and replenishment of these task forces is a critical problem with which naval planners are faced. It is vital for ships at sea to be provided with a reliable flow of supplies for them to maintain a continuous readiness posture.

Forces afloat are presently supplied with fuel, food, ammunition, and other bulk goods primarily by large supply ships which accompany task forces during their operations and may periodically proceed to bases ashore to replenish themselves. Transfer of goods at sea is largely accomplished by combatants coming alongside these support ships and receiving supplies by use of various rigs. The nature of the primary alongside method of transfer and its consequent grouping of numbers of ships over a period of time increases the vulnerability of all participating vessels. Therefore, time of replenishment of the whole combatant task force is of primary concern. The problem of scheduling a replenishment operation of a combatant task force by a group of supply ships such as oilers, ammunition ships, and other support vessels has been studied by McCullough [3], Gordon and Copes [2], Patterson [5], and Waggoner [6]. Waggoner built a mathematical model to study the scheduling of a fixed number of combatants by two supply ships and computed the distribution of total replenishment time of all combatants. It is the purpose of this paper to consider the analogous problem with three supply ships.

II. FORMULATION OF THE PROBLEM

In general the problem may be considered from a queueing theory approach with each supply ship representing a service facility and each combatant a customer. It may or may not be necessary for each customer to visit each facility, depending upon the needs of the customer and the number of duplicate facilities. Waggoner [6] and Milch and Waggoner [4] employ a random walk model for study of a two facility queue which operates in parallel and series simultaneously with a finite number of customers. Such a model is applicable to the replenishment situation. It is the purpose of this paper to extend the random walk model to the case of three supply ships under the following assumptions:

(i) A finite number of combatants are initially present, and each must be served once by all three supply ships.

(ii) The service times at each supply ship are independent, exponential random variables with parameters λ, μ, and ν for ships I, II, and III, respectively, and each one serves only one combatant at a time.

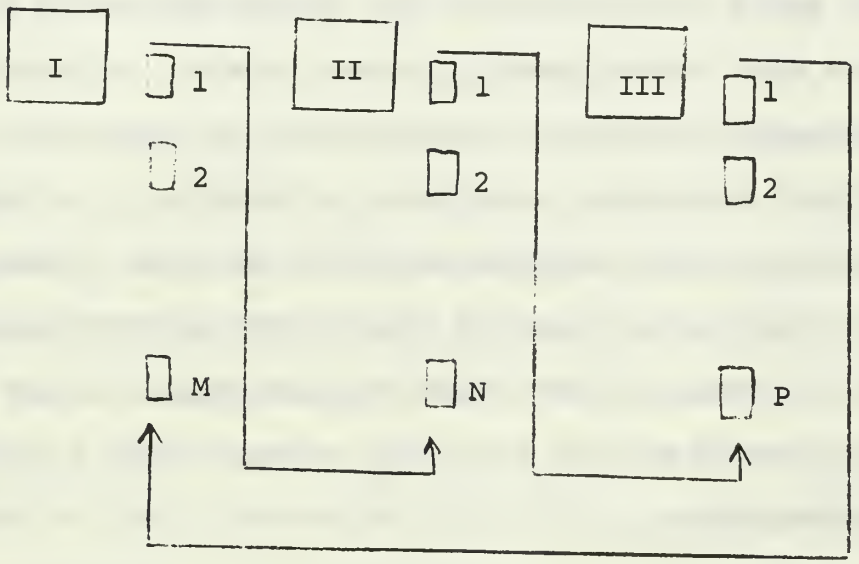
(iii) Service is simultaneous and consecutive. When a combatant has been replenished in one queue he moves to the next queue in cyclic order. Each facility continues in operation until it has served all combatants.

(iv) No additional combatants join the queues from outside the system after the commencement of operations.

(v) When a combatant has been served by all three supply ships he departs the system.

(vi) Transit times between queues are considered negligible.

Figure 1 illustrates the initial positions of all vessels and the order of flow from one queue to another.



$$M + N + P = L$$

Figure 1

III. THE THREE DIMENSIONAL MODEL

Milch and Waggoner [4] have represented an analagous two facility queueing system by a random walk in a two dimensional lattice. The random walk begins at the origin at time zero and proceeds by horizontal and vertical steps to the point (L,L) where L = M + N, and M and N are the initial number of customers in queues I and II, respectively. A horizontal step corresponds to completion of servicing in one queue, and a vertical step is a completed service in the other. Howevera all paths to (L,L) are not possible since if X(t) and Y(t) represent the amount of customers served in each respective queue at time t, then $X(t) \leq M + Y(t)$ and $Y(t) \leq N + X(t)$. Figure 2 shows a path in such a representation.

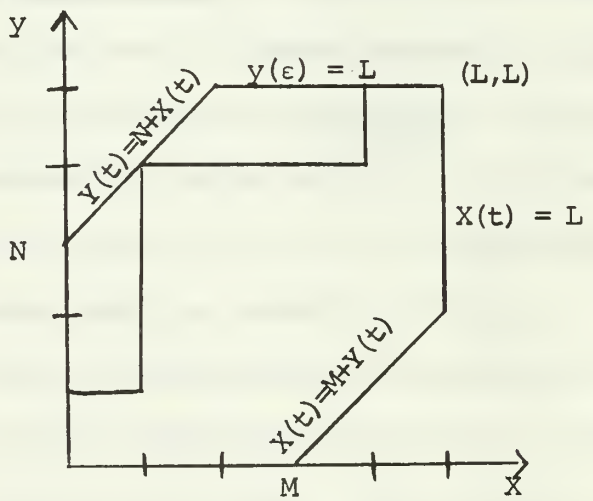


Figure 2

In the two dimensional model the distribution of total operation time, T, is found in terms of its Laplace transform.

If three supply ships are present as in Figure 1 a three dimensional random walk may be formulated which begins at the origin at time zero and proceeds by unit steps at a time in one of the three axial directions to the point (L,L,L) where $L = M + N + P$, and M, N, P are the initial number of ships in the first, second, and third queue, respectively. All paths to (L,L,L) do not represent a replenishment operation, however, since if $X(t)$, $Y(t)$, and $Z(t)$ are the number of ships served in queues I, II, and III respectively, then $X(t) \leq M + Z(t)$, $Y(t) \leq N + X(t)$, and $Z(t) \leq P + Y(t)$ are constraints that must hold during the whole operation. These constraints define boundary planes for the three dimensional lattice.

The random walk paths are further bounded by $X(t) = L$, $Y(t) = L$, $Z(t) = L$. For the purpose of clarity in further development these planes and their intersections will be catagorized as follows:

Boundary	Constituents
A	$x = z + M$, $z = L$, and their intersection
B	$y = x + N$, $y = L$, and their intersection
C	$z = y + P$, $z = L$, and their intersection
D	Intersection of A and B
E	Intersection of A and C
F	Intersection of B and C

Note: D is not regarded as part of A or B. Similar statements may be made about E and F.

IV. DISTRIBUTION OF TOTAL REPLENISHMENT TIME

A. PATH PROBABILITIES

The first move of the random walk is made one unit in an axial direction with probabilities $\lambda/\lambda + \mu + \nu$, $\mu/\lambda + \mu + \nu$, and $\nu/\lambda + \mu + \nu$ for axes X, Y, and Z, respectively, since these are the probabilities of stations I, II, or III, respectively, completing replenishment of the first combatant under the assumption of independent, exponential service times. Due to the memoriless property of the exponential distribution these probabilities are true at all steps along the random walk, except at points on the boundaries whose probabilities are as follows:

Boundary	Axial Direction	Probability
A	Y	$\mu/\mu + \nu$
	Z	$\nu/\mu + \nu$
B	X	$\lambda/\lambda + \nu$
	Z	$\nu/\lambda + \nu$
C	X	$\lambda/\lambda + \mu$
	Y	$\mu/\lambda + \mu$
D	Z	one
E	Y	one
F	X	one

The total time for the replenishment is the time needed for the random walk to proceed from the origin to the point (L,L,L) which is the sum of the times needed to make each of the 3 L steps required for the walk. The distribution of time for any of the steps not originating

on a boundary is the minimum of three independent, exponential distributions and therefore is itself exponential with parameter $\lambda + \mu + \nu$. Similarly the duration of steps originating on boundaries A, B, and C is exponential with parameters $\mu + \nu$, $\lambda + \nu$, and $\lambda + \mu$, respectively. The distribution of steps originating on boundaries D, E, and F is exponential with parameters ν, μ, and λ respectively.

Let $Q(i,j,k,\ell,m,n)$ be a route that has exactly i, j, k, ℓ, m, n positions on boundaries A, B, C, D, E, F, respectively. Then $Q(i,j,k,\ell,m,n)$ has $3L - (i + j + k + \ell + m + n)$ positions along the path that are not on a boundary. The total replenishment time, T, is composed of the following random variables, given that $Q(i,j,k,\ell,m,n)$ occurs:

$$T = A_i + B_j + C_k + D_\ell + E_m + F_n + G_g \qquad (4\text{-}1)$$

where A_i is the sum of i steps each of which originates on boundary A, and B_j, C_k, D_ℓ, E_m, and F_n are similarly defined. G_g is the sum of g steps none of which originates on any boundary where $g = 3L - i - j - k - \ell - m - n$.

Path $Q(i,j,k,\ell,m,n)$ contains L steps in each of the three axial directions. If the L steps in the x direction are examined, exactly n of them will begin on boundary F each with probability one. Some number, j_1, will begin on boundary B each with probability $\lambda/\lambda + \nu$ and some number, k_1, begin on boundary C each with probability $\lambda/\lambda + \mu$. The remaining $L - n - j_1 - k_1$ steps will begin from a non-boundary position each with probability $\lambda/\lambda + \mu + \nu$. Similar statements may be made concerning the steps in the other two axial directions, and the results are tabulated below.

Axis Direction	Number of Steps	Step Probability
x	n	one
	j_1	$\lambda/\lambda + \nu$
	k_1	$\lambda/\lambda + \mu$
	$L - (n + j_1 + k_1)$	$\lambda/\lambda + \mu + \nu$
y	m	one
	i_1	$\mu/\mu + \nu$
	k_2	$\mu/\lambda + \mu$
	$L - (m + i_1 + k_2)$	$\mu/\lambda + \mu + \nu$
z	ℓ	one
	i_2	$\nu/\mu + \nu$
	j_2	$\nu/\lambda + \nu$
	$L - (\ell + i_2 + j_2)$	$\nu/\lambda + \mu + \nu$

where $i = i_1 + i_2$, $j = j_1 + j_2$, $k = k_1 + k_2$. Then

$$P(Q) = r(i,j,k,\ell,m,n) \left(\frac{\lambda}{\lambda+\mu+\nu}\right)^{L-(n+j_1+k_1)} \left(\frac{\lambda}{\lambda+\nu}\right)^{j_1}$$

$$\left(\frac{\lambda}{\lambda+\mu}\right)^{k_1} \left(\frac{\mu}{\lambda+\mu+\nu}\right)^{L-(m+j_1+k_2)} \left(\frac{\mu}{\mu+\nu}\right)^{j_1} \left(\frac{\mu}{\lambda+\mu}\right)^{k_2}$$

$$\left(\frac{\nu}{\lambda+\mu+\nu}\right)^{L-(\ell+i_2+j_2)} \left(\frac{\nu}{\mu+\nu}\right)^{i_2} \left(\frac{\nu}{\lambda+\nu}\right)^{j_2} \qquad (4\text{-}2)$$

$$P(Q) = r(i,j,k,\ell,m,n) \left(\frac{\lambda+\mu+\nu}{\mu+\nu}\right)^{i} \left(\frac{\lambda+\mu+\nu}{\lambda+\nu}\right)^{j}$$

$$\left(\frac{\lambda+\mu+\nu}{\lambda+\mu}\right)^{k} \left(\frac{\nu}{\lambda+\mu+\nu}\right)^{L-\ell} \left(\frac{\mu}{\lambda+\mu+\nu}\right)^{L-m} \left(\frac{\lambda}{\lambda+\mu+\nu}\right)^{L-n}$$

where $Q = Q(i,j,k,\ell,m,n)$ and $r(i,j,k,\ell,m,n)$ is the number of paths from the origin to the point (L,L,L) which have i,j,k,ℓ,m,n points in common with boundaries A, B, C, D, E, F, respectively.

B. TRANSFORM OF THE DISTRIBUTION

From equation (4-1) the following Laplace transform can be tabulated:

Random Variable	Laplace Transform
A_i	$(\mu+\nu/\mu+\nu+s)^i$
B_j	$(\lambda+\nu/\lambda+\nu+s)^j$
C_k	$(\lambda+\mu/\lambda+\mu+s)^k$
D_ℓ	$(\nu/\nu+s)^\ell$
E_m	$(\mu/\mu+s)^m$
F_n	$(\lambda/\lambda+s)^n$
G_g	$(\lambda+\mu+\nu/\lambda+\mu+\nu+s)^g$

The Laplace transform of the total replenishment time, T, is:

$$f^*_T(s) = E(e^{-sT}) = \sum_{\substack{i,j,k \\ \ell,m,n}} E(e^{-sT}|Q)\ p(Q)$$

$$f^*_T(s) = \sum_{\substack{i,j,k, \\ \ell,m,n}} r(i,j,k,\ell,m,n)\ \left(\frac{\lambda\mu\nu}{\lambda+\mu+\nu}\right)^L \left(\frac{\lambda+\mu+\nu}{\lambda+\mu+\nu+s}\right)^{3L}$$

$$\left(\frac{\lambda+\mu+\nu+s}{\mu+\nu+s}\right)^i \left(\frac{\lambda+\mu+\nu+s}{\lambda+\nu+s}\right)^j \left(\frac{\lambda+\mu+\nu+s}{\lambda+\mu+s}\right)^k \qquad (4\text{-}3)$$

$$\left(\frac{\lambda+\mu+\nu+s}{\nu+s}\right)^\ell \left(\frac{\lambda+\mu+\nu+s}{\mu+s}\right)^m \left(\frac{\lambda+\mu+\nu+s}{\lambda+s}\right)^n$$

V. COUNTING TECHNIQUES

A. DISCUSSION

An expression for $r(i,j,k,\ell,m,n)$ has not been developed in this paper. However, this chapter demonstrates several counting techniques for the two supply ship, two dimensional random walk which may be useful in the eventual determination of $r(i,j,k,\ell,m,n)$. Additionally, for the three dimensional model an expression is found for the number of paths proceeding from the origin to (u,v,w) which touch no boundaries. In what follows, the binomial coefficient, $\binom{n}{r} = n!/r!(n-r)!$

B. TWO DIMENSIONAL COUNTING

1. The Number of Paths Not Touching Boundaries

First a method will be formulated for counting the number of paths proceeding from (0,0) to (u,v) which do not touch or cross either boundary $y = x + a$, $(a > 0)$, or boundary $y = x + b$, $(b < 0)$. This number is equal to the total $\binom{u+v}{u}$ paths going to (u,v) less the number of paths which touch or cross either boundary. By the Reflection Principle (see page 70 of [1]), the number touching or crossing $y=x+a$ is $\binom{u+v}{v-a}$ and the number touching or crossing $y = x + b$ is $\binom{u+v}{v-b}$. Since in both of these quantities paths touching or crossing both boundaries are included, the number of such paths must be added back. This quantity may be computed by dividing it into two mutually exclusive categories -- those paths which touch or cross boundary $y = x + a$ first, and those paths which touch or cross boundary $y = x + b$ first. Such paths will be counted by a double application of the Reflection Principle. In Figure 3 consider path 1 which falls into the first of the two categories.

The first reflection is with respect to line $y = x + a$. Both the boundary $y = x + b$ and that part of path 1 beyond its first common point with $y = x + a$ are reflected. The reflected path, path 1', ends in point $(v-a, u+a)$. The common points of path 1 and $y = x + b$ are reflected into common points between path 1' and the line $y = x + 2a-b$ (this being the reflection of line $y = x + b$). Therefore, the number of paths in the first category is equal to the number of paths ending in $(v-a, u+a)$ and touching or crossing the line $y = x + 2a-b$. The latter number is

$$\binom{v-a+u+a}{u+a-(2a-b)} = \binom{u+v}{u-a+b} \tag{5-1}$$

by a second application of the Reflection Principle. Similarly the number of paths going to (u,v) after touching $y = x + b$ first is $\binom{u+v}{u+a-b}$. Such a path is path 2 in Figure 3. Therefore the number of paths to (u,v) touching neither $y = x + a$ nor $y = x + b$ is

$$\binom{u+v}{u} - \binom{u+v}{u+a} - \binom{u+v}{u+b} + \binom{u+v}{u+a-b} + \binom{u+v}{u-a+b} \tag{5-2}$$

By extending the boundary reflection employed in the preceding paragraph it is possible to compute the number of paths proceeding to (u,v) which touch or cross $y = x + a$, $a > 0$, first, $y = x + b$, $b < 0$, second, and $y = x + c$, $c > a$, third, before going to (u,v). In Figure 4 since $y = x + a$ is the first boundary to be struck, reflect $y = x + b$ and $y = x + c$ with respect to $y = x + a$ yielding $y = x + 2a-b$ and $y = x + 2a-c$, respectively. Since $y = x + 2a-b$ is the next boundary to be struck by the reflected path, reflect $y = x + 2a-c$ with respect to $y = x + 2a-b$, yielding $y = x+2a - 2b+c$. The twice reflected path

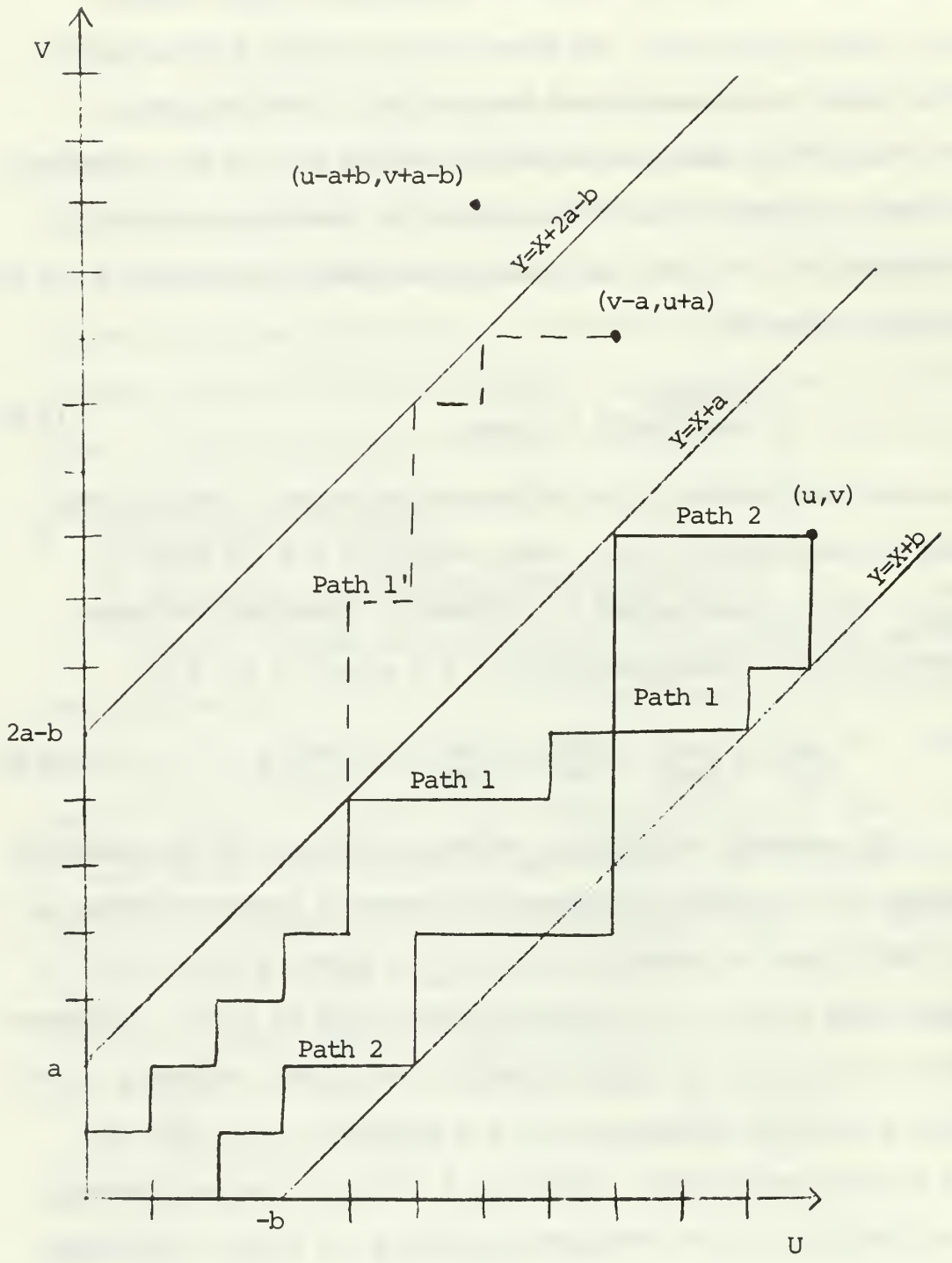

V
(u-a+b,v+a-b)
Y=X+2a-b
(v-a,u+a)
Y=X+a
(u,v)
Path 2
Y=X+b
Path 1'
Path 1
2a-b
Path 1
Path 2
a
-b
U

Figure 3

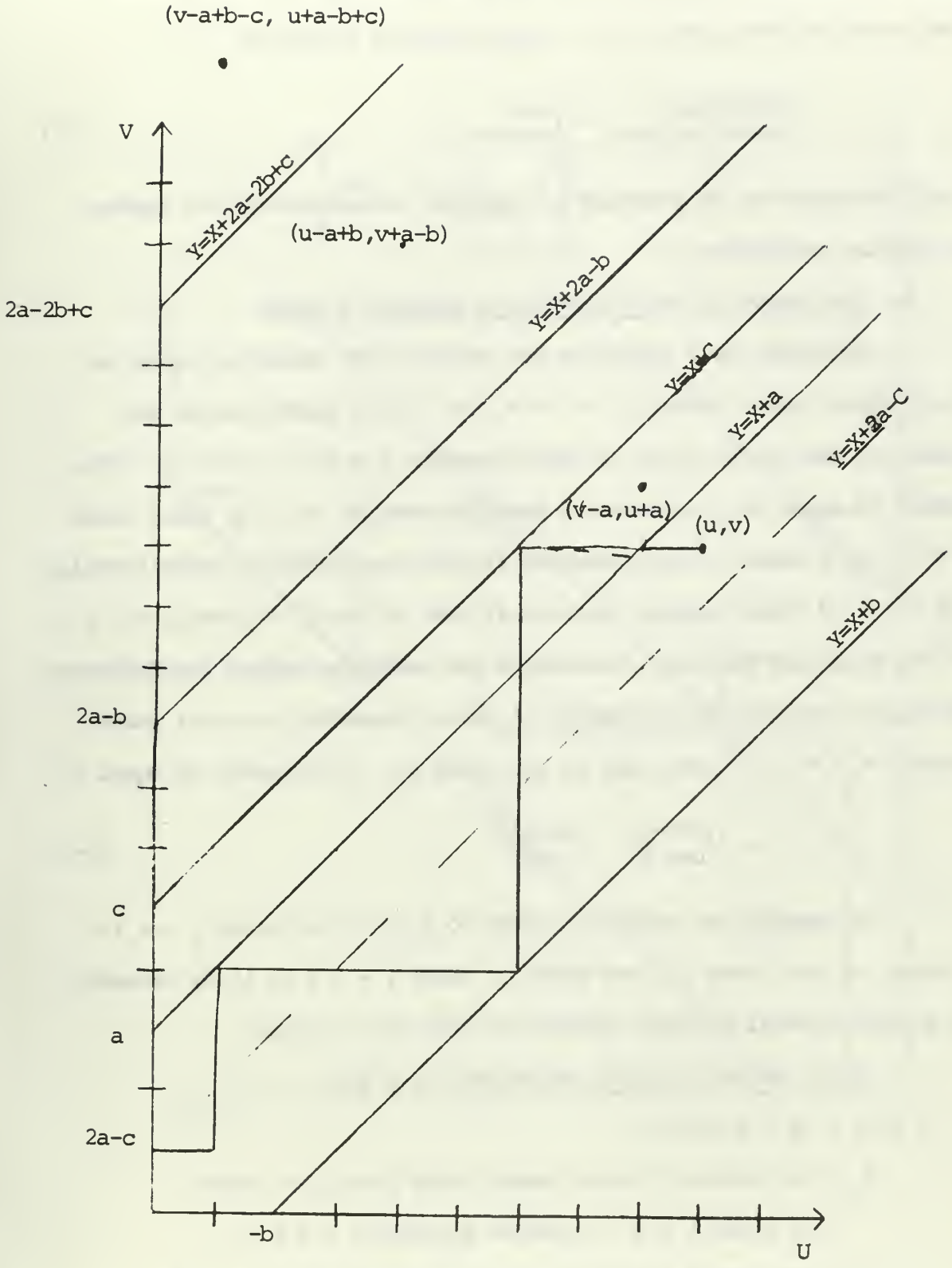

(v-a+b-c, u+a-b+c)
V
Y=X+2a-2b+c
(u-a+b,v+a-b)
2a-2b+c
Y=X+2a-b
Y=X+c
Y=X+a
Y=X+2a-c
(v-a,u+a)
(u,v)
Y=X+b
2a-b
c
a
2a-c
-b
U

Figure 4

will then strike $y = x+2a - 2b+c$ and then proceed to $(u-a+b, v+a-b)$. The number of such paths is, by the Reflection Principle,

$$\binom{u-a+b+v+a-b}{v+a-b-(2a-2b+c)} = \binom{u+v}{v-a+b-c} \tag{5-3}$$

This technique may be extended to multiple reflections of any number of similar boundaries.

2. The Number of Paths Touching a Boundary d Times

Consider now a technique for counting the number of paths to (u,v) which touch boundary $y = x + a$, $(a > 0)$, d times, but do not cross it, and do not touch or cross boundary $y = x + b$, $(b < 0)$. This number is equal to the number of paths proceeding to (u,v) which touch $y = x + a$ d times without crossing it less the number of paths touching $y = x + a$ d times without crossing it that do touch or cross $y = x + b$. In [4] Milch and Waggoner developed a new technique called the Telescope Principle for counting the number of paths proceeding to (u,v) which touch $y + x + a$ d times and do not cross it. This number is equal to

$$\binom{u+v-d}{u+a-1} - \binom{u+v-d}{u+a} \tag{5-4}$$

To compute the number of paths to (u,v) which touch $y = x + a$ d times, do not cross it, and touch or cross $y = x + b$, it is necessary to examine several disjoint subsets of that set of paths.

If the number of paths in the set is N then

$N = N_1 + N_2 + N_3$ where

N_1 = the number of paths among these paths that touch or cross $y = x + b$ before touching $y = x + a$

N_2 = the number of paths among them that touch $y = x + a$ d times without crossing it before touching or crossing $y = x + b$

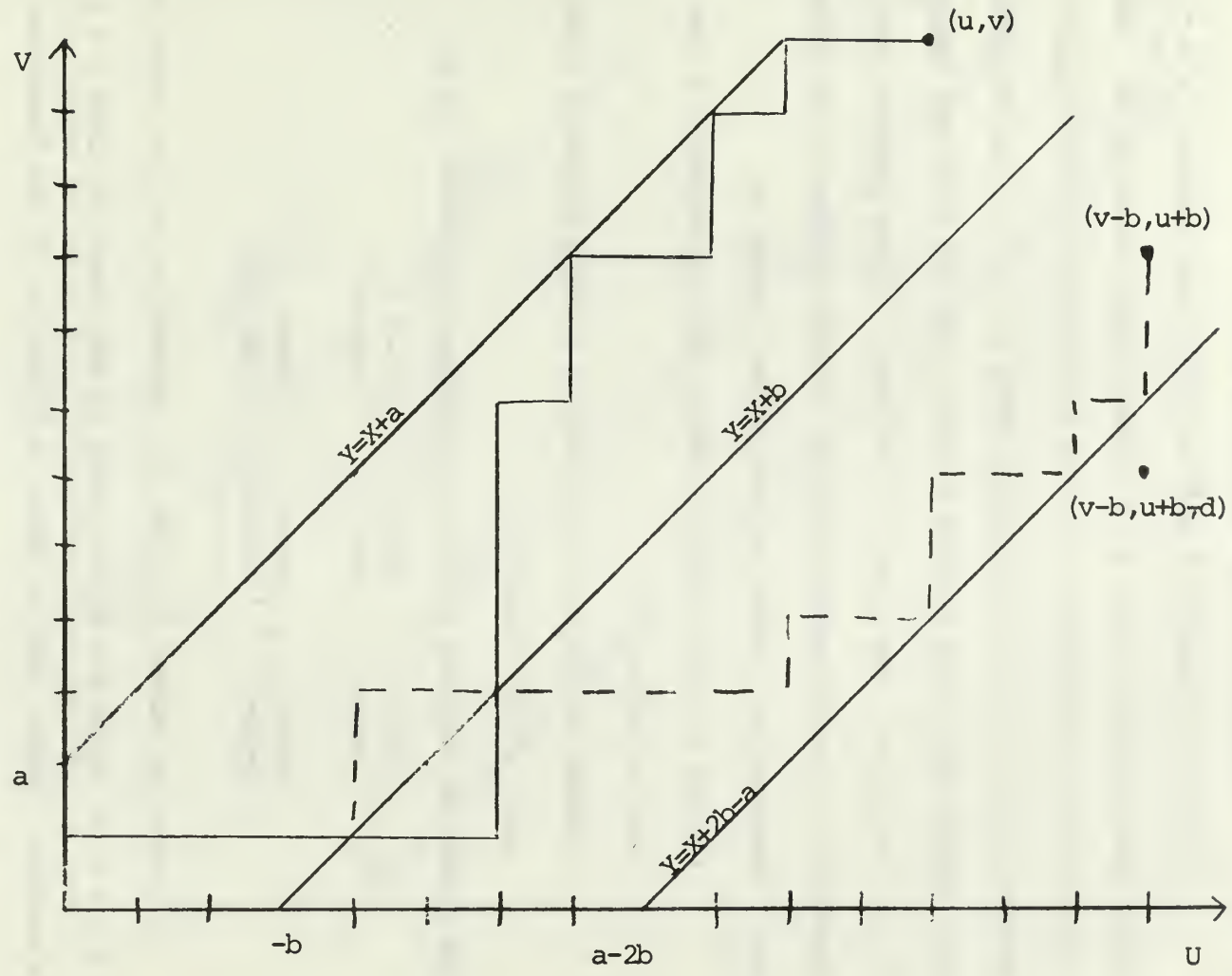


Figure 5

$$N_3 = \sum_{i=1}^{d-1}$$ (the number of paths among them that touch $y = x + a$ exactly i times and do not cross it before touching or crossing $y = x + b$)

To find N_1 consider a path to (u,v) that crosses $y = x + b$ first, then touches $y = x + a$ d times without crossing it. Such a path is shown in Figure 5. If the path beyond its first point of contact with $y = x + b$ is reflected about $y = x + b$, the reflected path (dotted line) will proceed to (v-b, u+b) and touch the reflection of $y = x + a$, i.e., the line $y = x + 2b-a$, d times without ever crossing it. The Telescope Principle is then applied to the reflected path. Therefore the number of such paths is equal to

[number of paths to (v-b, u+b-d) hitting neither $y = x+a$ nor $y = x+2b-a-d$]

- [number of paths to (v-b, u+b-d) hitting neither $y = x+a$ nor $y = x+2b - a-d+1$]

This number is obtained from (5-2) yielding the formula for N_1:

$$N_1 = [\binom{u+v-d}{u+a-b-1} - \binom{u+v-d}{u+a-b}] - [\binom{u+v-d}{u+2a-b-1} - \binom{u+v-d}{u+2a-b}] + [\binom{u+v-d}{u-2a+3b} - \binom{u+v-d}{u-2a+3b+1}] \quad (5\text{-}5)$$

Next the number of paths, N_2, proceeding to (u,v) which touch $y = x + a$ d times but do not cross it then touch or cross $y = x + b$ is counted. Such a path is shown in Figure 6. If the Telescope Principle is applied to this path about boundary $y = x + a$, its equivalent path (dashed line) can be seen to meet the following criteria:

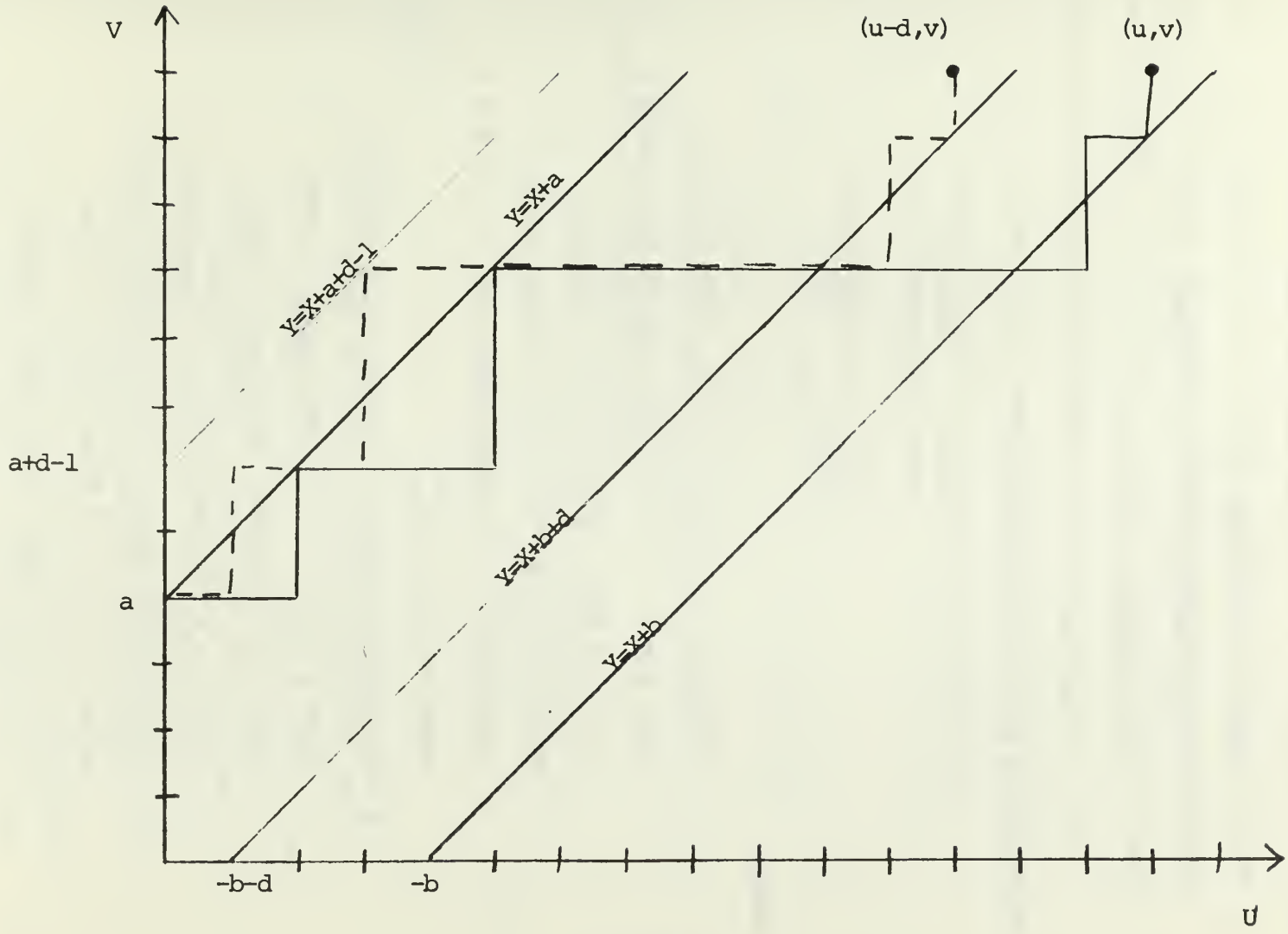


Figure 6

(i) it must touch but not cross line
$y = x + a + d-1$ first and then must
touch or cross line $y = x + b + d$.
It may touch $y = x + a + d-1$ again
after touching $y = x + b$.

(ii) it must go to $(u-d,v)$.

The number of paths so constrained is

$$N_2 = A - B - C + D = [\binom{u+v-d}{v+a-b-1} - \binom{u+v-d}{v+a-b}] - [\binom{u+v-d}{u+2a-b-1} - \binom{u+v-d}{u+2a-b}] \quad (5\text{-}6)$$

where

A = the number of paths to $(u-d,v)$ which touch or cross $y = x + a + d-1$ first, then touch or cross $y = x + b + d$. From (5-1), $A = \binom{u+v-d}{v+a-b-1}$.

B = the number of paths to $(u-d,v)$ which touch or cross $y = x + a + d$ first, then touch $y = x + b + d$. From (5-1), $B = \binom{u+v-d}{v+a-b}$.

C = the number of paths to $(u-d,v)$ which touch or cross $y = x + a + d-1$ first, touch or cross $y = x + b + d$ second, then touch or cross $y = x + a + d$. From (5-3), $C = \binom{u+v-d}{v-2a+b-d+1}$.

D = the number of paths to $(u-d,v)$ which touch or cross $y = x + a + d$ first, touch or cross $y = x + b+d$ second, then touch or cross $y = x + a+d$ again. D must be added since it has been subtracted in both B and C. From (5-3), $D = \binom{u+v-d}{v-2a+b-d}$.

Those paths which proceed to (u,v) and touch $y = x + a$ (d-1) times before touching $y = x + b$, touch $y = x + a$ a total of d times, and do not touch $y = x + a + 1$ are counted next. Figure 7 shows a path of this type. Applying the Telescope Principle to this path about boundary $y = x + a$ its equivalent path (dashed line) can be seen to be constrained as follows:

(i) it must touch boundary $y = x + a + d-2$ first, touch $y = x + b + d-1$ second, and then touch $y = x + a + d-1$.

(ii) it must go to (u-d,v).

(iii) it may not touch $y = x + a + d-1$ until it has touched $y = x + b + d-1$.

(iv) it may not touch boundary $y = x + a + d$.

The number of such paths is equal to

$$A' - B' - C' + D' = \binom{u+v-d}{u+2a-b-2} - \binom{u+v-d}{u+2a-b-1} - \left[\binom{u+v-d}{u+2a-b-1} - \binom{u+v-d}{u+2a-b}\right] \qquad (5\text{-}7)$$

where

A' = the number of paths to (u-d,v) which must touch $y = x+a+d-2$ first, touch $y = x+b+d-1$ second, then touch $y = x+a+d-1$. From (5-3), $A' = \binom{u+v-d}{v-2a+b-d+2}$

B' = the number of paths to (u-d,v) which must touch $y = x+a+d-1$ first, touch $y = x+b+d-1$ second, then touch $y = x+a+d-1$. From (5-3), $B' = \binom{u+v-d}{v-2a+b-d+1}$.

C' = the number of paths to (u-d,v) which must touch $y = x+a+d-2$ first, touch $y = x+b+d-1$ second, then touch $y = x+a+d$. From (5-3), $C' = \binom{u+v-d}{v-2a+b-d+1}$.

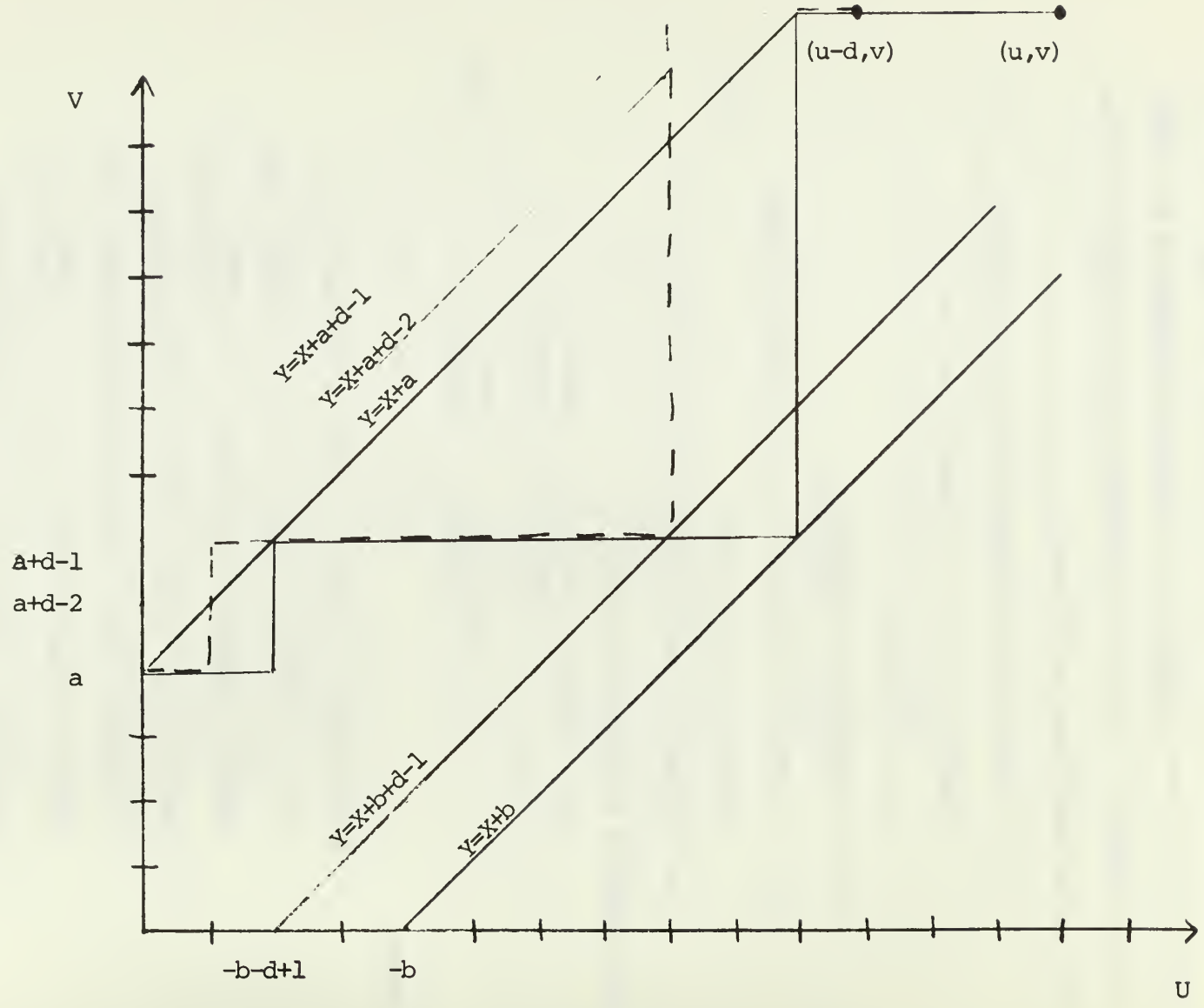


Figure 7

D' = the number of paths to (u-d,v) which must touch y=x+a+d-1 first, touch y=x+b+d-1 second, then touch y=x+a+d. D' must be added since it has been subtracted in both B' and C'. From (5-3), $D' = \binom{u+v-d}{v-2a+b-d}$.

When the numbers of those paths proceeding to (u,v) which touch y=x+a 1,2,...,(d-2) times, respectively, before touching y=x+b, touch y=x+a a total of d times, and do not touch y=x+a+1 are computed similarly, each is found to be equal to (5-7). Therefore the total number of paths in N_3 is

$$N_3 = (d-1)\binom{u+v-d}{u+2a-b-2} - 2(d-1)\binom{u+v-d}{u+2a-b-1} + (d-1)\binom{u+v-d}{u+2a-b}$$

This may be written as

$$N_3 = (d-1)\left[\binom{u+v-d}{u+2a-b-2} - \binom{u+v-d}{u+2a-b-1}\right]$$

$$- (d-1)\left[\binom{u+v-d}{u+2a-b-1} - \binom{u+v-d}{u+2a-b}\right] \qquad (5\text{-}8)$$

Now $N = N_1 + N_2 + N_3$, and from (5-5), (5-6), and (5-8)

$$N = \binom{u+v-d}{u+a-b-1} - \binom{u+v-d}{u+a-b} + \binom{u+v-d}{u-2a+3b} - \binom{u+v-d}{u-2a+3b+1}$$

$$+ \binom{u+v-d}{v+a-b-1} - \binom{u+v-d}{v+a-b} - (d+1)\left[\binom{u+v-d}{u+2a-b-1} - \binom{u+v-d}{u+2a-b}\right] \qquad (5\text{-}9)$$

$$+ (d-1)\left[\binom{u+v-d}{u+2a-b-2} - \binom{u+v-d}{u+2a-b-1}\right]$$

Combining the results of (5-4) and (5-9), the number of paths to (u,v) which touch boundary y=x+a d times, do not cross it, and do not touch boundary y=x+b is

$$[\binom{u+v-d}{u+a-1} - \binom{u+v-d}{u+a}] - [\binom{u+v-d}{u+a-b-1} - \binom{u+v-d}{u+a-b}]$$

$$-[\binom{u+v-d}{u-2a+3b} - \binom{u+v-d}{u-2a+3b+1}] - [\binom{u+v-d}{v+a-b-1} - \binom{u+v-d}{v+a-b}] \qquad (5\text{-}10)$$

$$+ d\,[\binom{u+v-d}{u+2a-b-1} - \binom{u+v-d}{u+2a-b}] - (d-1)\,[\binom{u+v-d}{u+2a-b-2} - \binom{u+v-d}{u+2a-b-1}]$$

C. THREE DIMENSIONAL COUNTING

The number of paths from the origin to some point (u,v,w) of the three supply ship model which do not touch any of the boundaries of Chapter III will be counted next. This number is equal to the total number of paths from the origin to (u,v,w), irrespective of boundaries, less the number of paths which touch at least one of the boundaries. The total number of paths from the origin to (u,v,w) irrespective of any boundaries is the number of possible combinations of u steps in the x direction, v steps in the y direction and w steps in the z direction. This number is the multinomial coefficient, $\binom{u+v+w}{u,v} = \binom{u+v+w}{u,v,w} = (u+v+w)!/u!v!w!$

In order to count the number of paths from (0,0,0) to (u,v,w) which contact boundary planes, it will be necessary to extend the Reflection Principle to three dimensions. By the Reflection Principle $\binom{u+v+w}{w+a,v,u-a}$, $\binom{u+v+w}{v-a,u+a,w}$, and $\binom{u+v+w}{u,w-a,v+a}$ are the number of paths that have at least one common point with the planes x=z+a, y=x+a, and z=y+a, respectively, among the $\binom{u+v+w}{u,v,w}$ paths connecting the origin and the point (u,v,w).

If N' is the number of paths from the origin to (u,v,w) which touch at least one boundary plane then

$$N' = E' + F' + G' - H' - I' - J' + K'$$

where

E' = the number of paths touching $x = z + M$

F' = the number of paths touching $y = x + N$

G' = the number of paths touching $z = y + P$

H' = the number of paths touching $x = z + M$ and $y = x + N$

I' = the number of paths touching $x = z + M$ and $z = y + P$

J' = the number of paths touching $y = x + N$ and $z = y + P$

K' = the number of paths touching $x = z + M$, $y = x + N$, and $z = y + P$.

By direct application of the Reflection Principle

$$E' = \binom{u+v+w}{w+M,v,u-M}$$

$$F' = \binom{u+v+w}{v-N,u+N,w}$$

$$G' = \binom{u+v+w}{u,w-P,v+P}$$

H' is the sum of two numbers; the number of paths which strike $x = z + M$ first and the number striking $y = x + N$ first. I' and J' are the sum of similar terms. The formula for J' is explained below and H' and I' are listed in tabular form. The reflection of the plane $y = x + N$ about the plane $z = y + P$ is $x = z - (N + P)$. Any path proceeding to the point (u,v,w), initially touching the plane $z = y + P$ (one or more times), and then proceeding to the plane $y = x + N$, is reflected beyond its first point of contact with $z = y + P$ about that plane. The reflected path touches or crosses the reflection of $y = x + N$, i.e., the plane $x = z - (N + P)$, and proceeds to the point $[(u-N), (v-P), (w + N + P)]$. The number of such paths is $\binom{u+v+w}{u-N,v-P}$.

The reflection of plane $z = y + P$ about $y = x + N$ is $x = z - (N + P)$. Any path proceeding to the point (u,v,w), initially touching the plane $y = x + N$ (one or more times), and then proceeding to the plane $z = y + P$

is reflected beyond its first point of contact with $y = x + N$ about that plane. The reflected path touches or crosses the reflection of $z = y + P$, i.e., the plane $x = z - (N + P)$, and proceeds to the point $[(u - (N + P)), (v + N), (w + P)]$. Therefore $J' = \binom{u+v+w}{u-N,v-P} + \binom{u+v+w}{v+N,w+P}$. H' and I' are similarly computed and are tabulated below.

Table for H'

Initial Plane	$y=x+N$	$x=z+M$
Second Plane	$x+z+M$	$y=x+N$
Reflected Plane	$z=y-(M+N)$	$z=y-(M+N)$
Reflected Point	$[u-N,v+M+N,w-M]$	$[u+M,v+N,w-(M+N]$
Number of Paths	$\binom{u+v+w}{u-N,w-M}$	$\binom{u+v+w}{u+M,v+N}$

$$H' = \binom{u+v+w}{u-N,w-M} + \binom{u+v+w}{u+M,v+N}$$

Table for I'

Initial Plane	$z=y+P$	$x=z+M$
Second Plane	$x=z+M$	$z=y+P$
Reflected Plane	$y=x-(M+P)$	$y=x-(M+P)$
Reflected Point	$[u+M,v-(M+P),v+P]$	$[u+M+P,v-P,w-M]$
Number of Paths	$\binom{u+v+w}{u+M,w+P}$	$\binom{u+v+w}{v-P,w-N}$

$$I' = \binom{u+v+w}{u+M,w+P} + \binom{u+v+w}{v-P,w-M}$$

K may be computed in similar fashion by permiting all three boundary planes and applying the Reflection Principle twice after each permutation.

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13. ABSTRACT

The operation of replenishment at sea is investigated for three supply ships and L combatants using queueing theory concepts and a random walk model in three dimensions. The distribution for total replenishment time given the initial number of combatants to be replenished by each supply ship and a specified cyclic order of replenishment is expressed in terms of its Laplace transform under the assumption of independent, exponential service times for each supply vessel. A method for counting all possible sequences of replenishment is not found, but some preliminary counting techniques are developed which may be useful in its eventual determination.

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